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Steady Flow of Thin Liquid Films on an Inclined Solid surface

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Abstract: - In this paper, we consider the thinning process of an inclined thin liquid film over a solid boundary with an inclination angleθ to the horizontal in gravity driven flow. Throughout this work, we assumed that the fluid thickness is constant far behind the front and we neglect the thickness of the film at the beginning of the motion. The differential equation of the film thickness is obtained analytically and the solution of equation that represents the film thickness is obtained numerically by using Rung-Kutta method.

Keywords: - *Thin Liquid Films , Navier-Stokes equations , continuity equation .*

I. INTRODUCTION

We present here some of the theoretical aspects of the instability development in an inclined thin liquid films on a solid surface in two dimensional coordinate system with an inclination angle θ to the horizontal . There are different types of phenomena that can occur, such as drainage, details of rupture, non–Newtonian surface properties in moving contract lines in thin liquid films [1]. These phenomena can help to describe the physical processes that occur in our real world. [2] have studied the case of contact line instabilities of thin liquid films but with constant flux configuration and also they considered some global models of a moving contact lines. [3] studied the thin liquid films flowing down the inverted substrate in three dimensional flow. [4] investigated the dynamics of an inclined thin liquid films of variable thickness in steady and unsteady cases and when the film is stationary and uniform. [5] considered the stability of thin liquid films and sessile droplets. The stability of the contact line of thin fluid film flows with constant flux configuration is considered by [6]. [7] considered the spreading of thin liquid films with small surface tension in the case when the flow is unsteady. [8] have studied the drainage of thin liquid films on an inclined solid surfacefor unsteadynflow by using similarity solution. In this paper we investigate the drainage of the inclined thin liquid films. The solution of the governing equations of the liquid film thickness is obtained numerically.

II. GOVERNING EQUATIONS:

Let $q=(u,w)$ denotes the fluid velocity, where u and w are the velocity components in x and z directions respectively. Let $z=h(x,t)$ be the equation of the inclined thin liquid films as shown in Figure (1) and the flow is in x direction.
The continuity equation is given by:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \tag{1}$$

and from the incompressibility condition, we have

$$w=-z\frac{\partial u}{\partial x} \tag{2}$$

and this insures that $\frac{\partial u}{\partial x}$ is a function of x only.
The Navier-Stokes equations in x and z directions respectively for an inclined thin liquid film are given by:

$$\rho\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g\sin\theta \tag{3}$$

And

$$\rho\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g\cos\theta \tag{4}$$

where ρ, μ and P are the density, viscosity of fluid and P the pressure. In lubrication theory the inertia terms can be neglected and the Navier-Stokes equations (3) and (4) become

$$\frac{\partial P}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g\sin\theta \tag{5}$$

and $$\frac{\partial P}{\partial z} = \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - \rho g \cos\theta \tag{6}$$

III. BOUNDARY CONDITIONS

The boundary conditions to be imposed on the bounding surfaces are as follows:

III.1 The no-slip condition gives

at $z = 0$, $u = 0$ (7)

III. The shear stress condition on the surface vanishes, that is:

at $z = h, \ \frac{\partial u}{\partial z} = 0$ (8)

Furthermore the normal stress condition on the surface of the film is given by

$$\overline{w} = -\sigma k \tag{9}$$

Where σ is the surface tension and k is the curvature of the surface which is given by

$$k = \frac{\partial^2 h}{\partial x^2} / ((1 + (\frac{\partial h}{\partial x})^2)^{3/2} \tag{10}$$

The pressure is related to the normal stress by the formula

$$P = \overline{w} + 2\mu(\frac{\partial w}{\partial z})_s \tag{11}$$

Now from the continuity equation (1), we have

$$(\frac{\partial w}{\partial z})_s = -(\frac{\partial u}{\partial x})_s , \tag{12}$$

and thus equation (11) is then become

$$P = \overline{w} - 2\mu(\frac{\partial u}{\partial x})_s \tag{13}$$

since $(\frac{\partial h}{\partial x})^2$ is so small for thin liquid films, thus equation (10) is then reduces to give

$$k = \frac{\partial^2 h}{\partial x^2},$$

and equation (13) is then become

$$P = -\sigma\frac{\partial^2 h}{\partial x^2} - 2\mu(\frac{\partial u}{\partial x})_s \tag{14}$$

Differentiate equation (14), we get

$$\frac{\partial P}{\partial x} = -\sigma\frac{\partial^3 h}{\partial x^3} - 2\mu(\frac{\partial^2 u}{\partial x^2})_s . \tag{15}$$

It is to be noted that $\frac{\partial P}{\partial x}$ is a function of x only on the surface of the liquid film z = h (x).

Now from the conservation of mass and since the free surface is a stream line, the derivative following the motion (the material or the substantial derivative) $\frac{DF}{Dt}$ must be vanished on z = h (x) and thus, for steady flow we have

$$u\frac{\partial F}{\partial x} + w\frac{\partial F}{\partial z} = 0 \tag{16}$$

Where

$$F(x,z) = z - h(x) \tag{17}$$

from equations (16) and (17), we have

$$u\frac{\partial h}{\partial x} - w = 0 \tag{18}$$

now substitute equation (2) into equation (18), we obtain

$$u\frac{\partial h}{\partial x} + z\frac{\partial u}{\partial x} = 0 \tag{19}$$

on the surface of the liquid film z = h(x), equation (19) then gives

$$u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \tag{20}$$

IV. Dimensional analysis:

We now introduce the following non-dimensional variables as follow:

$$x = L\bar{x}\ ,\ z = H\bar{z}\ ,\ u = U\bar{u}\ ,\ w = \varepsilon U\bar{w}\ ,\ P = \frac{MU}{L}\bar{P}\ , \tag{21}$$

$h = H\bar{h}$ and $\varepsilon = \frac{H}{L} << 1$

where L, H and U are the characteristics.

By using the dimensionless variables the continuity equation (1) then gives

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0 \tag{22}$$

Also equations (14) in non-dimensional variables has the form

$$\bar{P} = -\frac{\varepsilon}{Ca}\frac{\partial^2 \bar{h}}{\partial \bar{x}^2} - 2\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_s \tag{23}$$

Where $Ca = \frac{\mu U}{\sigma}$ is the capillary number.

Furthermore the Navier-Stokes equations (5) and (6) in non-dimensional variables are respectively given by

$$\frac{\partial \bar{P}}{\partial \bar{x}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{2}{\varepsilon^2}\frac{\partial^2 \bar{u}}{\partial \bar{z}^2} + \frac{B}{Ca}\sin\theta \tag{24}$$

and $$\frac{\partial \bar{P}}{\partial \bar{z}} = \varepsilon^2\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{z}^2} - \frac{B}{Ca}\cos\theta \tag{25}$$

Where $B = \frac{\rho g L^2}{\sigma}$ is the bond number.

Also equation (20) has the following non-dimensional form

$$\bar{u}\frac{\partial \bar{h}}{\partial \bar{x}} + \bar{h}\frac{\partial \bar{u}}{\partial \bar{x}} = 0$$

or

$$\frac{\partial}{\partial \bar{x}}(\bar{u}\bar{h}) = 0$$

or

$$(\bar{u}\bar{h}) = \mathrm{Re}\,\nu \tag{26}$$

Where

$\mathrm{Re} = \frac{\rho UL}{\mu}$ is the Reynolds number and

$\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity.

From (23) and on the surface of the film, we have

$$\frac{\partial \bar{P}}{\partial \bar{x}} = -\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3} - 2\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s \tag{27}$$

From equations (24) and (27), we get

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{2}{\varepsilon^2}\frac{\partial^2 \bar{u}}{\partial \bar{z}^2} + \frac{B}{Ca}\sin\theta = -\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3} - 2\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s \tag{28}$$

Or

$$\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3} + 3\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s + \frac{2}{\varepsilon^2}\frac{\partial^2 \bar{u}}{\partial \bar{z}^2} + \frac{B}{Ca}\sin\theta = 0 \tag{29}$$

Integrating equation (29) with respect to $\bar{Z}$, we get

$$\left(\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3} + 3\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s + \frac{B}{Ca}\sin\theta\right)\bar{z} + \frac{2}{\varepsilon^2}\frac{\partial \bar{u}}{\partial \bar{z}} = A_1(\bar{x}) \tag{30}$$

The boundary conditions (7) and (8) in non- dimensional variables have the form:

At $\bar{z}=0 \quad , \bar{u}=0$ (31)

At $\bar{z}=\bar{h}(\bar{x}) \quad , \quad \frac{\partial \bar{u}}{\partial \bar{z}}=0$ (32)

By using the boundary condition (32), equation (30) reduces to give

$$\left(\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3}+3\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s+\frac{B}{Ca}\sin\theta\right)\bar{h} = A_1(\bar{x}) \tag{33}$$

And thus equation (30), gives

$$\left(\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3}+3\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s+\frac{B}{Ca}\sin\theta\right)(\bar{z}-\bar{h})+\frac{2}{\varepsilon^2}\frac{\partial \bar{u}}{\partial \bar{z}}=0 \tag{34}$$

Integrating equation (34), we get

$$\left(\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3}+3\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s+\frac{B}{Ca}\sin\theta\right)\left(\frac{\bar{z}^2}{2}-\bar{h}\bar{z}\right)+\frac{2}{\varepsilon^2}\bar{u} = A_2(\bar{x}) \tag{35}$$

From the boundary condition (31) and equation (35), we have

$A_2(\bar{x})=0,$

And thus equation (35), then gives

$$\left(\frac{\varepsilon}{Ca}\frac{\partial^3 \bar{h}}{\partial \bar{x}^3}+3\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s+\frac{B}{Ca}\sin\theta\right)\left(\frac{\bar{z}^2}{2}-\bar{h}\bar{z}\right)+\frac{2}{\varepsilon^2}\bar{u} = 0 \tag{36}$$

Since the first term in equation (36) is very small so we can neglected and thus equation (36) reduces to give

$$\left[3\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s+\frac{B}{Ca}\sin\theta\right]\left(\frac{\bar{z}^2}{2}-\bar{h}\bar{z}\right)+\frac{2}{\varepsilon^2}\bar{u} = 0 \tag{37}$$

From equation (26), we have

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right)_s = -\operatorname{Re}\upsilon\left(\frac{\bar{h}^2\bar{h}''-2\bar{h}'^2\bar{h}}{\bar{h}^3}\right) \tag{38}$$

Substituting equation (38) into equation (37), we get

$$\left[-3\operatorname{Re}\upsilon(\bar{h}^2\bar{h}''-2\bar{h}'^2\bar{h})+\frac{B}{Ca}\sin\theta\,\bar{h}^3\right]\left(\frac{\bar{z}^2}{2}-\bar{h}\bar{z}\right)+\frac{2}{\varepsilon^2}\bar{u}\,\bar{h}^3=0 \tag{39}$$

From (26) and equation (39), we obtain

$$\left[-3\operatorname{Re}\upsilon(\bar{h}^2\bar{h}''-2\bar{h}'^2\bar{h})+\frac{B}{Ca}\sin\theta\,\bar{h}^3\right]\left(\frac{\bar{z}^2}{2}-\bar{h}\bar{z}\right)+\frac{2\operatorname{Re}\upsilon}{\varepsilon^2}\bar{h}^2=0 \tag{40}$$

On the surface of the film z = h(x), equation (40) then gives

$$\left[-3\operatorname{Re}\upsilon(\bar{h}^2\bar{h}''-2\bar{h}'^2\bar{h})+\frac{B}{Ca}\sin\theta\,\bar{h}^3\right]\left(-\frac{\bar{h}^2}{2}\right)+\frac{2\operatorname{Re}\upsilon}{\varepsilon^2}\bar{h}^2=0$$

Or

$$3\operatorname{Re}\upsilon(\bar{h}^2\bar{h}''-2\bar{h}'^2\bar{h})-\frac{B}{Ca}\sin\theta\,\bar{h}^3+\frac{4\operatorname{Re}\upsilon}{\varepsilon^2}=0 \tag{41}$$

Equation (41) represents the non-linear ordinary differential equation for the thickness of the inclined thin liquid film on a solid boundary in $(\bar{x},\bar{h})-$ plane for steady flow.

Also from (39), we have

$$\bar{u}=\left(\frac{3\varepsilon^2\operatorname{Re}\upsilon\,\bar{h}''}{2\bar{h}}-\frac{3\varepsilon^2\operatorname{Re}\upsilon\,\bar{h}'^2}{\bar{h}^2}-\frac{B\varepsilon^2}{2Ca}\sin\theta\right)\left(\frac{\bar{z}^2}{2}-\bar{h}\bar{z}\right) \tag{42}$$

And the average velocity over the film thickness is given by

$$\bar{u}_{av}=\frac{1}{h}\int_0^h \bar{u}\,dz \tag{43}$$

And thus from equations (42) and (43) we have

$$\bar{u}_{av}=-\frac{\varepsilon^2\operatorname{Re}\upsilon}{2}\bar{h}\bar{h}''+\varepsilon^2\operatorname{Re}\upsilon\,\bar{h}'^2+\frac{\varepsilon^2 B}{6Ca}\sin\theta\,\bar{h}^2 \tag{44}$$

Some solution curves of equation (41) are presented in figure (2)and shows that the thickness of liquid film for increases as the angle of inclination to the horizontal increases, moreover figure (3) show the thickness of some liquid films namely linseed oil, olive oil and glycerinfor θ=1.57.

V. FIGURES AND TABLES

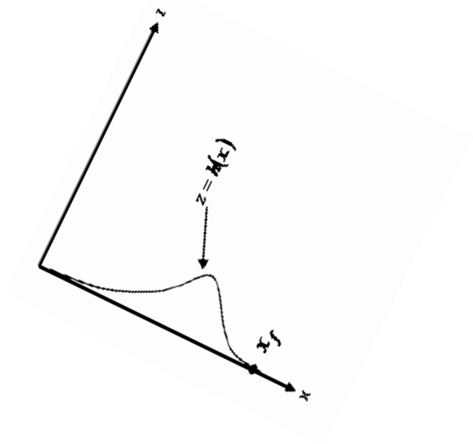
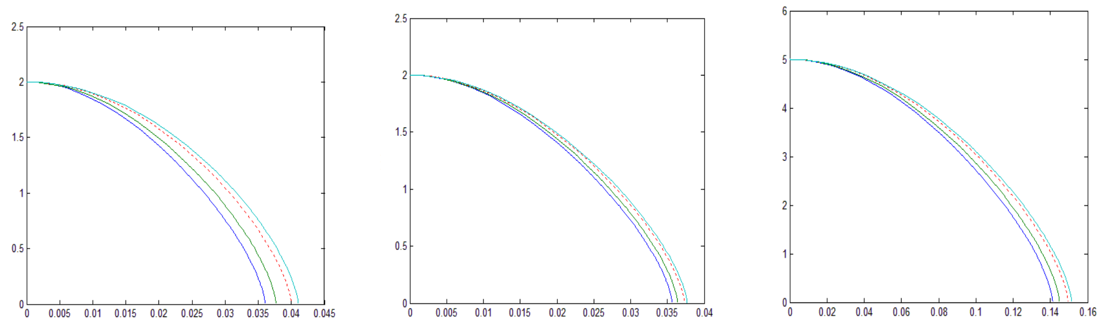


Fig. (1): Sketch of the flow in two–dimensional geometry, where the capillary ridge is just behind the flow front.



(water) (Glyserin) (Carbon Tetrachloride)

Fig.(2): Thickness of thin liquid films for different inclination angles

θ= 0.174 θ= 0.52 θ= 1.04 θ= 1.57

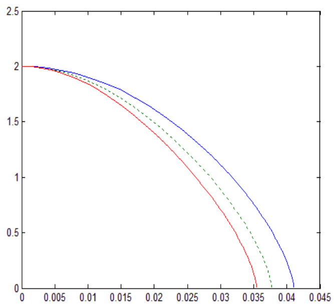


Fig.(3): Thickness of thin liquid films for linseed oil,olive oil and glycerin for θ= 1.57.

linseed oil olive oil glycerin

VI. CONCLUSION

The dynamics of a free surface liquid films flowing over an inclined solid boundary with an inclination angle to the horizontal is very useful in industrial coating and spinning processes. The non- linear differential equation that governs such flow for steady case in non- dimensional form is obtained and the solution curves for different inclination angles shows that the thickness of the thin liquid films increases as the inclination angle increases..

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